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Numerical Integration and Approximation of Differentiable Functions, II

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1. INTRODUCTION

Suppose we are given a finite sequence $x_1,..., x_N$ of numbers in the unit interval [0, 1] and a finite sequence $p_1,..., p_N$ of nonnegative numbers such that

$$\sum_{k=1}^{N} p_k = 1.$$

We call the numbers $p_1,..., p_N$ weights of the numbers $x_1,..., x_N$, respectively. Further, let

$$a_0 = 0 \qquad \text{and} \qquad a_k = \sum_{i=1}^{k} p_i, \qquad k = 1,..., N.$$

We shall consider in this paper quadrature formulae of the type

$$\int_0^1 f(x)\,dx = \sum_{k=1}^{N} \sum_{j=0}^{r} A_{kj} f^{(j)}(x_k) + R_N^{(r)}(f), \tag{1.1}$$

where the function f is r-times differentiable on [0, 1] and the coefficients A_{kj} are given by

$$A_{kj} = \frac{(a_k - x_k)^{j+1} - (a_{k-1} - x_k)^{j+1}}{(j+1)!}.$$

Quadrature formulae of the type (1.1) have been studied in [1–4] and

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similar ones have been studied in [5]. It is easy to see that if $r=0$, then the quadrature formula (1.1) can be written in the form

$$\int_0^1 f(x)\,dx = \sum_{k=1}^{N} p_k f(x_k) + R_N^{(0)}(f).$$

Consequently, (1.1) is a generalization of the general quadrature process with positive weights.

We shall also investigade in our paper approximation of r-times differentiable functions on $[0, 1]$ by means of functions $L_N^{(r)}(f; x)$ which are defined on $[0, 1]$ by

$$L_N^{(r)}(f; x) = \sum_{j=0}^{r} \frac{f^{(j)}(x_k)}{j!}(x-x_k)^j \qquad \text{if} \quad x \in [a_{k-1}, a_k),\ 1 \leqslant k \leqslant N. \tag{1.2}$$

(We shall assume in what follows that $[a, b) = [a, b]$ if $b = 1$.)

In the present paper, we obtain some new upper bounds for the error of the quadrature formulae of the type (1.1) and for the error of the approximation of differentiable functions by means of functions of the type (1.2). Our results improve or generalize almost all results which have been obtained in [1–4, 6]. The main results of the paper have been announced in [7].

2. Notation and Definitions

For a sequence $x_1,..., x_N$ in $[0, 1]$ with a weight sequence $p_1,..., p_N$, we define the functions g and h on $[0, 1]$ by

$$g(x) = x - \sum_{\substack{1 \leqslant k \leqslant N \\ x_k < x}} p_k$$

and

$$h(x) = |x - x_k| \qquad \text{if} \quad x \in [a_{k-1}, a_k),\ 1 \leqslant k \leqslant N.$$

Then the number

$$D_N = \|g\|_C = \sup_{0 \leqslant x \leqslant 1} |g(x)|$$

is said to be the (extreme) discrepancy of $x_1,..., x_N$ with respect to the weights $p_1,..., p_N$. Further, the number

$$D_N^{(p)} = \|g\|_{L_p} = \left(\int_0^1 |g(x)|^p\,dx\right)^{1/p}, \qquad 0 < p < \infty,$$

is called the L^p discrepancy of $x_1, \dots, x_N$ with respect to the weights $p_1, \dots, p_N$. It is easy to see that if $p_1 = p_2 = \cdots = p_N = 1/N$, then D_N and $D_N^{(p)}$ are equal to the extreme and L^p discrepancies of $x_1, \dots, x_N$, respectively (see, e.g., [8, pp. 90, 97]).

As a measure of the distribution of $x_1, \dots, x_N$ with respect to the weights $p_1, \dots, p_N$, except the discrepancies D_N and $D_N^{(p)}$, we also use the number

$$\Delta_N^{(p)} = \left(\int_0^1 |g(x)|^{p+1}\, dx\right) \Big/ \left(\int_0^1 |g(x)|^p\, dx\right), \qquad 0 \leqslant p < \infty.$$

Obviously, $\Delta_N^{(0)} = D_N^{(1)}$.

As a characteristic of a function f defined on $[0, 1]$, we use its modulus of continuity

$$\omega(f; \delta) = \sup\{|f(x') - f(x'')|: |x' - x''| \leqslant \delta,\ x',\ y' \in [0, 1]\}$$

or its modulus of smoothess

$$\tau(f; \delta)_{L_p} = \|\omega(f; x; \delta)\|_{L_p}, \qquad 0 < p < \infty,$$

where

$$\omega(f; x; \delta) = \sup\{|(x') - f(x'')|: x', x'' \in [x - \delta/2, x + \delta/2] \cap [0, 1]\}.$$

For the history of modulus $\tau(f; \delta)_{L_p}$ and its properties see [9].

Let us recall that a continuous function $\omega(\delta)$ defined on $[0, +\infty)$ is called a modulus of continuity if $\omega(0) = 0$ and

$$0 \leqslant \omega(\delta_2) - \omega(\delta_1) \leqslant \omega(\delta_2 - \delta_1) \qquad \text{for} \quad 0 \leqslant \delta_1 \leqslant \delta_2. \tag{2.1}$$

Now, we shall introduce some classes of functions. Let C, B, and M denote the set of all continuous functions on $[0, 1]$, the set of all bounded functions on $[0, 1]$, and the set of all bounded and measurable functions on $[0, 1]$, respectively. Further, for a modulus of continuity $\omega(\delta)$, we denote by H^ω the set of all functions f for which the inequality

$$\omega(f; \delta) \leqslant \omega(\delta)$$

holds for every $\delta \in [0, +\infty)$. If

$$\omega(\delta) = C\delta^\alpha,$$

where $0 < \alpha \leqslant 1$ and C is an absolute constant, we shall write $H_\alpha(C)$ instead of H^ω. Finally, for a natural r ($r \in \mathbb{N}$) we put

$$W^r\mathfrak{M} = \{f: f^{(r)} \in \mathfrak{M}\},$$

where $\mathfrak{M}$ is an arbitrary set of functions which are defined on $[0, 1]$. We also put $W^0\mathfrak{M} = \mathfrak{M}$.

3. Auxiliary Lemmas

We need the following lemmas.

Lemma 1. *The discrepancies D_N, $D_N^{(p)}$, and $\Delta_N^{(p)}$ are related by the inequality*

$$D_N^{(p+1)} < \Delta_N^{(p)} < D_N \qquad \text{for} \quad 0 < p < \infty. \tag{3.1}$$

Proof. The second inequality in (3.1) is obvious. It is easy to show that the first inequality in (3.1) is equivalent to the inequality

$$\|g\|_{L_p} < \|g\|_{L_{p+1}},$$

which holds for every $p > 0$. The lemma is proved.

Lemma 2 (Proinov [6]). *Let $x_1 \leqslant \cdots \leqslant x_N$. Then for $0 < p < \infty$, we have*

$$D_N^{(p)} = \|h\|_{L_p}$$

and

$$D_N = \|h\|_C = \max_{1 \leqslant k \leqslant N} \max\{|x_k - a_{k-1}|, |x_k - a_k|\}.$$

Lemma 3 (see, e.g., [10, Sect. 2, Problem 75]). *Let $f(x)$ and $p(x)$ be Riemann integrable functions on $[a, b]$ such that*

$$\int_a^b p(x)\,dx > 0$$

and inequalities

$$m \leqslant f(x) \leqslant M \qquad \text{and} \qquad p(x) \geqslant 0$$

hold for every $x \in [a, b]$, where m and M are absolute constants. Then if $\varphi(x)$ is a concave function on $[m, M]$, we have

$$\frac{\int_a^b p(x)\,\varphi(f(x))\,dx}{\int_a^b p(x)\,dx} \leqslant \varphi\left(\frac{\int_a^b p(x)\,f(x)\,dx}{\int_a^b p(x)\,dx}\right).$$

LEMMA 4. *Let $\tilde{h}$ be a function defined on* $[0, 1]$ *by*

$$\tilde{h}(x) = c_k + C\,|x - x_k| \qquad \textit{if} \quad x \in [a_{k-1}, a_k),\ 1 \leqslant \kappa \leqslant N, \tag{3.2}$$

where the numbers $c_1, \ldots, c_N$ are given by

$$c_1 = 0, \qquad c_{k+1} = c_k + C\,|a_k - x_k| - C\,|a_k - x_{k+1}|, \qquad k = 1, \ldots, N, \tag{3.3}$$

and C is an absolute constant. Then

$$\tilde{h} \in H_1(C).$$

Proof. Since $a_k \in [a_k, a_{k+1})$, it follows from (3.2) and (3.3) that

$$\tilde{h}(a_k) = c_{k+1} + C\,|a_k - x_{k+1}| = c_k + C\,|a_k - x_k|.$$

Therefore,

$$\tilde{h}(x) = c_k + C\,|x - x_k| \qquad \textit{if} \quad x \in [a_{k-1}, a_k],\ 1 \leqslant k \leqslant N. \tag{3.4}$$

Let $x', x'' \in [a_{k-1}, a_k)$, where $1 \leqslant k \leqslant N$. Then from (3.4), we deduce

$$\begin{aligned} |\tilde{h}(x') - \tilde{h}(x'')| &= C\, \big|\, |x' - x_k| - |x'' - x_k|\, \big| \\ &\leqslant C\,|(x' - x_k) - (x'' - x_k)| \leqslant C\,|x' - x''|. \end{aligned} \tag{3.5}$$

Now let $x' \in [a_{i-1}, a_i)$ and $x'' \in [a_{j-1}, a_j)$, where $1 \leqslant i < j \leqslant N$. Then from (3.5), we get

$$\begin{aligned} &|\tilde{h}(x') - \tilde{h}(x'')| \\ &\quad = |[\tilde{h}(x') - \tilde{h}(a_i)] + [\tilde{h}(a_i) - \tilde{h}(a_{i+1})] + \cdots + [\tilde{h}(a_{j-1}) - \tilde{h}(x'')] \\ &\quad \leqslant C(|x' - a_i| + |a_i - a_{i+1}| + \cdots + |a_{j-1} - x''|) \\ &\quad = x'' - x' = |x' - x''|. \end{aligned}$$

Consequently,

$$\omega(\tilde{h}, \delta) \leqslant C\delta$$

for $\delta \geqslant 0$. This completes the proof of the lemma.

COROLLARY 1. *Let the sequences $x_1, \ldots, x_N$ and $p_1, \ldots, p_N$ be related by*

$$a_k = \frac{x_k + x_{k+1}}{2}, \qquad k = 1, \ldots, N. \tag{3.6}$$

Then

$$h \in H_1(1).$$

Proof. If (3.6) holds, then from (3.3) it follows that

$$c_1 = c_2 = \cdots = c_N = 0.$$

Therefore,

$$h(x) = \tilde{h}(x)$$

with $C = 1$. Now Corollary 1 follows from Lemma 4.

LEMMA 5. *Suppose* (3.6) *holds and we are given a function* $\omega(\delta)$ *defined on* $[0, 1]$ *which satisfies* (2.1). *Put*

$$f(x) = \omega(h(x))$$

for $x \in [0, 1]$. *Then*

$$\begin{aligned} \omega(f; \delta) &= \omega(\delta) \qquad \textit{for} \quad 0 \leqslant \delta \leqslant D_N, \\ &= \omega(D_N) \qquad \textit{for} \quad \delta \geqslant D_N. \end{aligned}$$

Proof. Let $\delta \geqslant 0$. We proved in Corollary 1 that $h \in H_1(C)$. Hence, it follows from (2.1) that

$$\begin{aligned} |f(x') - f(x'')| &= \omega(h(x')) - \omega(h(x''))| \\ &\leqslant \omega(|h(x') - h(x'')|) \leqslant \omega(|x' - x''|) \leqslant \omega(\delta) \end{aligned} \tag{3.7}$$

for all $x', x'' \in [0, 1]$ with $|x' - x''| \leqslant \delta$. Suppose first that $0 \leqslant \delta \leqslant D_N$. It follows from Lemma 2 that there exists an integer k $(1 \leqslant k \leqslant N)$ such that either $D_N = x_k - a_k$ or $D_N = a_k - x_k$. We treat only the second alternative, the firts one being completely similar. Choose $x' = x_k$ and $x'' = x_k + \delta$. Since $a_{k-1} \leqslant x_k \leqslant x_k + \delta \leqslant x_k + D_N = a_k$, it follows that $x', x'' \in [a_{k-1}, a_k]$. Now by the definition of h, we obtain

$$|f(x') - f(x'')| = \omega(\delta). \tag{3.8}$$

From (3.7) and (3.8), we conclude that

$$\omega(f; \delta) = \omega(\delta) \qquad \text{for} \quad 0 \leqslant \delta \leqslant D_N. \tag{3.9}$$

Now let $\delta \geqslant D_N$. Then from (3.9) and Lemma 2, we have

$$\omega(D_N) = \omega(f; D_N) \leqslant \omega(f; \delta) \leqslant \|f\|_C = \omega(\|g\|_C) = \omega(D_N),$$

which means that

$$\omega(f;\delta)=\omega(D_N) \qquad \text{for} \quad \delta\geqslant D_N.$$

Lemma 5 is proved.

The following lemma belongs to S. B. Stečkin.

LEMMA 6 (see, e.g., [11, p. 182]). *For every modulus of continuity* $\omega(\delta)$ *there exists a concave modulus of continuity* $\omega^*(\delta)$ *such that*

$$\omega(\delta)\leqslant\omega^*(\delta)\leqslant 2\omega(\delta) \qquad \textit{for} \quad \delta\geqslant 0.$$

4. APPROXIMATION OF FUNCTIONS

Now we are ready to give some new upper bounds for the error of the approximation of r-times differentiable functions by means of functions of the type (1.2). We shall suppose in what follows that

$$x_1\leqslant x_2\leqslant\cdots\leqslant x_N. \tag{4.1}$$

THEOREM 1. *Let* $r\in\mathbb{N}$, $0<p<\infty$. *Then for every function* $f\in W^rB$, *we have*

$$\|f-L_N^{(r)}(f)\|_{L_p}\leqslant\frac{(D_N^{(pr)})^r}{(r-1)!}\int_0^1(1-x)^{r-1}\omega(f^{(r)};x\,D_N)\,dx. \tag{4.2}$$

Proof. Using Taylor's formula with the remainder in integral form it can be proved [1] that for $x\in[a_{k-1},a_k)$, $1\leqslant k\leqslant N$, we have

$$\begin{aligned}&f(x)-L_N^{(r)}(f;x)\\&\quad=\frac{(x-x_k)^r}{(r-1)!}\int_0^1(1-t)^{r-1}[f^{(r)}(x_k+(x-x_k)t)-f^{(r)}(x_k)]\,dt.\end{aligned} \tag{4.3}$$

Therefore,

$$\begin{aligned}&\|f-L_N^{(r)}(f)\|_{L_p}\\&\quad=\left(\sum_{k=1}^N\int_{a_{k-1}}^{a_k}|f(x)-L_N^{(r)}(f;x)|^p\,dx\right)^{1/p}\\&\quad=\frac{1}{(r-1)!}\left(\sum_{k=1}^N\int_{a_{k-1}}^{a_k}|x-x_k|^{pr}\right.\\&\qquad\times\left.\left|\int_0^1(1-t)^{r-1}[f^{(r)}(x_k+(x-x_k)t)-f^{(r)}(x_k)]\,dt\right|^p dx\right)^{1/p}.\end{aligned} \tag{4.4}$$

From (4.4), we obtain the estimate

$$\|f-L_N^{(r)}(f)\|_{L_p}$$
$$\leqslant \frac{1}{(r-1)!}\left(\sum_{k=1}^{N}\int_{a_{k-1}}^{a_k}|x-x_k|^{pr}\right.$$
$$\left.\times\left(\int_0^1(1-t)^{r-1}\omega(f^{(r)};|x-x_k|\,t)\,dt\right)^p dx\right)^{1/p}$$
$$=\frac{1}{(r-1)!}\left(\int_0^1 h(x)^{pr}\left(\int_0^1(1-t)^{r-1}\omega(f^{(r)};th(x))\,dt\right)^p dx\right)^{1/p}. \tag{4.5}$$

From Lemma 2, it follows that for all $x, t\in[0, 1]$,

$$\omega(f^{(r)};th(x))\leqslant\omega(f^{(r)};t\,\|h\|_C)=\omega(f^{(r)};t\,D_N).$$

Hence, we get from (4.5),

$$\|f-L_N^{(r)}(f)\|_{L_p}\leqslant\frac{1}{(r-1)!}\left(\int_0^1 h(x)^{pr}\,dx\right)^{1/p}\int_0^1(1-t)^{r-1}\omega(f^{(r)};t\,D_N)\,dt.$$

From this and Lemma 2, we obtain (4.2). The theorem is proved.

Passing to the limit as $p\to\infty$ in the inequality (4.2) we obtain the following

COROLLARY 2 (Proinov and Kirov [3]). *Let $r\in\mathbb{N}$. Then for every function $f\in W^rB$, we have*

$$\|f-L_N^{(r)}(f)\|_C\leqslant\frac{D_N}{(r-1)!}\int_0^1(1-x)^{r-1}\omega(f^{(r)};x\,D_N)\,dx.$$

The next theorem complements Theorem 1.

THEOREM 2. *Let $r\in\mathbb{N}$ and $0<p\leqslant 1$. Then for every function $f\in W^rB$ such that its* rth *derivative $f^{(r)}$ has concave modulus of continuity, we have*

$$\|f-L_N^{(r)}(f)\|_{L_p}\leqslant\frac{(D_N^{(pr)})^r}{(r-1)!}\int_0^1(1-x)^{r-1}\omega(f^{(r)};x\,\Delta_N^{(pr)})\,dx. \tag{4.6}$$

Moreover, if $p=1$ then the estimate (4.6) *cannot be improved in the sense that there exists a function* (*satisfying the requirement of the theorem*) *for which the inequality* (4.6) *changes into equality.*

Proof. Let $f \in W^r B$ and $\omega(\delta) = \omega(f^{(r)}; \delta)$ be a concave function on $[0, 1]$. It is easy to prove that the function

$$\psi(x) = \int_0^1 (1-t)^{r-1} \omega(tx)\, dt$$

is also concave on $[0, 1]$; that is, the inequality

$$\frac{\psi(x') + \psi(x'')}{2} \leqslant \psi\left(\frac{x' + x''}{2}\right)$$

holds for all $x', x'' \in [0, 1]$. On the other hand, the function $y = x^p$ $(0 < p \leqslant 1)$ is concave and increasing on $[0, +\infty)$. Therefore, the function

$$\varphi(x) = \psi(x)^p$$

is concave on $[0, 1]$. Obviously, (4.5) can be written in the form

$$\|f - L_N^{(r)}(f)\|_{L_p} \leqslant \frac{1}{(r-1)!} \left(\int_0^1 h(x)^{pr} \varphi(h(x))\, dx\right)^{1/p}. \tag{4.7}$$

Applying Lemma 3 with $p(x) = h(x)^{pr}$ we get

$$\frac{\int_0^1 h(x)^{pr} \varphi(h(x))\, dx}{\int_0^1 h(x)^{pr}\, dx} \leqslant \varphi\left(\frac{\int_0^1 h(x)^{pr+1}\, dx}{\int_0^1 h(x)^{pr}\, dx}\right) = \varphi(\Delta_N^{(pr)}).$$

From this and Lemma 2, we obtain

$$\begin{aligned}\int_0^1 h(x)^{pr} \varphi(h(x))\, dx &\leqslant (D_N^{(pr)})^p\, \varphi(\Delta_N^{(pr)}) \\ &= \left(D_N^{(pr)} \int_0^1 (1-t)^{r-1} \omega(f^{(r)}; t\, \Delta_N^{(pr)})\, dt\right)^p \end{aligned} \tag{4.8}$$

The estimate (4.6) follows from (4.7) and (4.8).

Now let $p = 1$. We shall prove that the estimate (4.6) cannot be improved in this case. Let $\tilde{f}$ be a function defined on $[0, 1]$ by

$$\tilde{f}^{(r)}(x) = Cx, \tag{4.9}$$

where C is an absolute constant. Evidently,

$$\omega(\tilde{f}^{(r)}; \delta) = C\delta. \tag{4.10}$$

Therefore, $\tilde{f} \in W^r H_1(C) \subset W^r B$ and $\omega(\delta) = \omega(\tilde{f}^{(r)}; \delta)$ is a concave function on [0, 1]; that is, $\tilde{f}$ satisfies the requirement of the theorem. From (4.4), (4.10), and Lemma 2, we get

$$\begin{aligned}\|\tilde{f}-L_N^{(r)}(\tilde{f})\|_{L_p}&=\frac{C}{(r+1)!}\sum_{k=1}^{N}\int_{a_{k-1}}^{a_k}|x-x_k|^{r+1}\,dx\\&=\frac{C}{(r+1)!}\int_0^1 h(x)^{r+1}\,dx\\&=\frac{(D_N^{(r)})^r}{(r-1)!}\int_0^1(1-x)^{r-1}\omega(\tilde{f}^{(r)};x\,\Delta_N^{(r)})\,dx.\end{aligned}$$

Consequently, if $p=1$ then for $\tilde{f}$ the inequality (4.6) changes into equality. Thus Theorem 2 is proved.

The next theorem can be proved by using the same method of proof.

THEOREM 2′. *Let $r\in\mathbb{N}$, $0<p\leqslant 1$, and $\omega(\delta)$ be a concave modulus of continuity. Then*

$$\sup_{f\in W^rH^\omega}\|f-L_N^{(r)}(f)\|\leqslant\frac{(D_N^{(pr)})^r}{(r-1)!}\int_0^1(1-x)^{r-1}\omega(x\Delta_N^{(pr)})\,dx. \tag{4.11}$$

Besides, the inequality (4.11) *changes into equality if $p=1$ and $\omega(\delta)=C\delta$, where C is an absolute constant.*

THEOREM 3. *For every function $f\in C$ whose modulus of continuity is a concave function on* [0, 1], *we have*

$$\|f-L_N^{(r)}(f)\|_L\leqslant\omega(f;D_N^{(1)}). \tag{4.12}$$

Moreover, the estimate (4.12) *cannot be improved.*

Proof. It is obvious that

$$\|f-L_N^{(0)}(f)\|_L=\sum_{k=1}^{N}\int_{a_{k-1}}^{a_k}|f(x)-f(x_k)|\,dx. \tag{4.13}$$

Consequently,

$$\|f-L_N^{(0)}(f)\|_L\leqslant\int_0^1\omega(f;h(x))\,dx.$$

Now, applying Lemma 3 with $\varphi(x)=\omega(f;x)$ and $p(x)\equiv 1$ we obtain

$$\|f-L_N^{(0)}(f)\|_L\leqslant\omega\left(f;\int_0^1 h(x)\,dx\right)=\omega(f;D_N^{(1)}).$$

Further, let $\tilde{f}$ be a function defined on [0, 1], by

$$\tilde{f}(x) = Cx, \tag{4.14}$$

where C is an absolute constant. From (4.12), (4.14), and Lemma 2, we get

$$\|\tilde{f} - L_N^{(0)}(\tilde{f})\|_L = C \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} |x - x_k| \, dx$$

$$= C \int_0^1 h(x) \, dx = \omega(\tilde{f}; D_N^{(1)}).$$

This equality shows that the estimate (4.12) cannot be improved. Theorem 3 is established.

The next theorem follows similarly.

THEOREM 3′. *Let $\omega(\delta)$ be a concave modulus of continuity. Then*

$$\sup_{f \in H^\omega} \|f - L_N^{(0)}(f)\|_L \leqslant \omega(D_N^{(1)}). \tag{4.15}$$

Besides, the inequality (4.15) *changes into equality if $\omega(\delta) = C\delta$, where C is an absolute constant.*

We shall denote in what follows by $\Gamma(x)$ and $B(x, y)$ the gamma function and the beta function, respectively, that is,

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} \, dt \qquad \text{and} \qquad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt.$$

THEOREM 4. *Let $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 < p < \infty$ and $0 < \alpha \leqslant 1$. Then*

$$\sup_{f \in W^r H_\alpha(C)} \|f - L_N^{(r)}(f)\|_{L_p} \leqslant C \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+r+1)} (D_N^{(pr+p\alpha)})^{r+\alpha}. \tag{4.16}$$

Moreover, the inequality (4.16) *changes into equality if either $\alpha = 1$ or* (3.6) *holds.*

Proof. Let $r \in \mathbb{N}$ and $f \in W^r H_\alpha(C)$. Since

$$\omega(f^{(r)}; \delta) \leqslant C\delta^\alpha,$$

it follows from (4.5) that

$$\|f - L_N^{(r)}(f)\|_{L_p} \leqslant \frac{C}{(r-1)!} B(\alpha+1, r) \left(\int_0^1 h(x)^{pr+p\alpha} \, dx \right)^{1/p}. \tag{4.17}$$

From this and Lemma 2, we get

$$\|f - L_N^{(r)}(f)\|_{L_p} \leqslant \frac{C}{(r-1)!} B(\alpha+1, r)(D_N^{(pr+p\alpha)})^{r+\alpha}. \tag{4.18}$$

From (4.18) using the well-known equality

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

and taking into account that

$$\Gamma(r) = (r-1)!$$

we obtain

$$\|f - L_N^{(r)}(f)\|_{L_p} \leqslant C \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+r+1)} (D_N^{(pr+p\alpha)})^{r+\alpha}.$$

Hence, (4.16) holds for $r \in \mathbb{N}$. Suppose that $r=0$ and $f \in H_\alpha(C)$. Obviously, we have

$$\|f - L_N^{(0)}(f)\|_{L_p} = \left(\sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} |f(x) - f(x_k)|^p \, dx \right)^{1/p}. \tag{4.19}$$

Since $\omega(f; \delta) \leqslant C\delta$, it follows from (4.19) and Lemma 2 that

$$\begin{aligned} \|f - L_N^{(0)}(f)\|_{L_p} &\leqslant C \left(\sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} |x - x_k|^{p\alpha} \, dx \right)^{1/p} \\ &= C \left(\int_0^1 h(x)^{p\alpha} \, dx \right)^{1/p} = C(D_N^{(p\alpha)})^\alpha. \end{aligned}$$

From this, we get (4.16) for $r=0$.

Now let $\alpha = 1$. Let us consider again the function $\tilde{f}$ defined on $[0, 1]$ by (4.9). From (4.10), it follows that $\tilde{f} \in W^r H_1(C)$. If $r \in \mathbb{N}$, then from (4.4) and Lemma 2, we obtain

$$\|\tilde{f} - L_N^{(r)}(\tilde{f})\|_{L_p} = \frac{C}{(r+1)!} (D_N^{(pr+p)})^{r+1}. \tag{4.20}$$

If $r=0$, then from (4.19) and Lemma 2, we obtain

$$\|\tilde{f} - L_N^{(0)}(\tilde{f})\|_{L_p} = C D_N^{(p)}.$$

Hence, (4.20) holds for every $r \in \mathbb{N}_0$; that is, (4.16) changes into equality if $\alpha = 1$.

Finally, suppose that $0<\alpha<1$ and (3.6) holds. Let $\bar{f}$ be a function defined on [0, 1] by

$$\bar{f}^{(r)}(x)=Ch(x)^{\alpha}, \tag{4.21}$$

where C is an absolute constant. Then, it follows from Lemma 5 that

$$\begin{aligned}\omega(\bar{f}^{(r)};\delta)&=C\delta^{\alpha} \quad &&\text{for} \quad 0\leqslant\delta\leqslant D_N,\\ &=CD_N^{\alpha} \quad &&\text{for} \quad \delta\geqslant D_N.\end{aligned} \tag{4.22}$$

Therefore, $\bar{f}\in W^rH_\alpha(C)$. If $r\in\mathbb{N}$, then from (4.4) and Lemma 2, we get

$$\|\bar{f}-L_N^{(r)}(\bar{f})\|_{L_p}=C\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+r+1)}(D_N^{(pr+p\alpha)})^{r+\alpha}. \tag{4.23}$$

If $r=0$, then from (4.19), we get

$$\|\bar{f}-L_N^{(0)}(\bar{f})\|_{L_p}=C(D_N^{(p\alpha)})^{\alpha}.$$

Hence, (4.22) holds for every $r\in\mathbb{N}_0$. This means that (4.16) changes into equality if (3.6) holds. The theorem is proved.

Passing to the limit as $p\to\infty$ in (4.16) we obtain the following

COROLLARY 3. *Let* $r\in\mathbb{N}_0$ *and* $0<\alpha\leqslant 1$. *Then*

$$\sup_{f\in W^rH_\alpha(C)}\|f-L_N^{(r)}(f)\|_C\leqslant C\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+r+1)}(D_N)^{r+\alpha}. \tag{4.24}$$

Moreover, the inequality (4.24) *changes into equality if either* $\alpha=1$ *or* (3.6) *holds.*

THEOREM 5. *Let* $r\in\mathbb{N}$ *and* $0<p<\infty$. *Then for every function* $f\in W^rM$ *and for all positive numbers* q *and* s *with* $1/q+1/s=1/p$, *we have*

$$\|f-L_N^{(r)}(f)\|_{L_p}\leqslant\frac{(D_N^{(sr)})^r}{r!}\tau(f^{(r)};2D_N)_{L_q}. \tag{4.25}$$

Proof. For all $x,z\in[0,1]$, we have

$$|x-(x_k+(x-x_k)z)|=|x-x_k|\cdot|1-z|\leqslant|x-x_k|,$$

where $1\leqslant k\leqslant N$. Therefore

$$|f^{(r)}(x_k+(x-x_k)z)-f^{(r)}(x_k)|\leqslant\omega(f^{(r)};x;2\,|x-x_k|).$$

From this and (4.4) we get

$$\|f - L_N^{(r)}(f)\|_{L_p} \leqslant \frac{1}{r!}\left(\int_0^1 h(x)^{pr}\omega(f^{(r)}; x; 2h(x))\,dx\right)^{1/p} \tag{4.26}$$

for $r \in \mathbb{N}$. From (4.19), it follows that (4.26) holds for $r = 0$ as well. Let us choose $\bar{p} = s/p$ and $\bar{q} = q/p$. Then $1/\bar{p} + 1/\bar{q} = 1$. Now using Hölder's inequality for integrals and Lemma 2 we deduce

$$\begin{aligned}
&\int_0^1 h(x)^{pr}\omega(f^{(r)}; x; 2h(x))^p\,dx \\
&\leqslant \left(\int_0^1 h(x)^{p\bar{p}r}\,dx\right)^{1/\bar{p}} \left(\int_0^1 \omega(f^{(r)}; x; 2h(x))^{p\bar{q}}\,dx\right)^{1/\bar{q}} \\
&= \left(\int_0^1 h(x)^{sr}\,dx\right)^{p/s} \left(\int_0^1 \omega(f^{(r)}; x; 2h(x))^q\,dx\right)^{p/q} \\
&\leqslant (D_N^{(sr)})^{pr} \left(\int_0^1 \omega(f^{(r)}; x; 2D_N)^q\,dx\right)^{pq} \\
&= [(D_N^{(sr)})^r \tau(f^{(r)}; 2D_N)_{L_q}]^p.
\end{aligned}$$

From this and (4.26), we get (4.25). Theorem 5 is proved.

Passing to the limit as $s \to \infty$ in (4.25) we obtain the following

COROLLARY 4 (Proinov and Kirov [3]). *Let $r \in \mathbb{N}$ and $0 < p < \infty$. Then for every function $f \in W^rM$, we have*

$$\|f - L_N^{(r)}(f)\|_{L_p} \leqslant \frac{(D_N)^r}{r!}\tau(f^{(r)}; 2D_N)_{L_p}.$$

5. Numerical Integration

Now using the above results we shall give some new upper bounds for the error of the quadrature formulae of the type (1.1).

THEOREM 6. *Let $r \in \mathbb{N}$. Then for every function $f \in W^rB$, we have*

$$|R_N^{(r)}(f)| \leqslant \frac{(D_N^{(r)})^r}{(r-1)!}\int_0^1 (1-x)^{r-1}\omega(f^{(r)}; x\,D_N)\,dx. \tag{5.1}$$

Proof. It is easy to check that for $r \in \mathbb{N}$ the error of the quadrature formula (1.1) can be written in the form

$$R_N^{(r)}(f) = \int_0^1 (f(x) - L_N^{(r)}(f; x))\, dx. \tag{5.2}$$

Consequently,

$$|R_N^{(r)}(f)| \leqslant \|f - L_N^{(r)}(f)\|_L. \tag{5.3}$$

Now (5.1) follows from Theorem 1 and (5.3). The theorem is proved.

Remark 1. It follows from Lemma 1 that Theorem 6 improves Theorem 4 of [3] and Theorem 3 of [4].

In [12] we proved that for every $f \in C$

$$|R_N^{(0)}(f)| \leqslant \omega(f; D_N). \tag{5.4}$$

It should be noted that this estimate was first proved by H. Niederreiter [13], for the case $p_1 = p_2 = \cdots = p_N = 1/N$.

From Theorem 6 and (5.4), we get the following

COROLLARY 5. *Let $r \in \mathbb{N}_0$. Then for every function $f \in W^rC$, we have*

$$|R_N^{(r)}(f)| \leqslant \frac{(D_N^{(r)})^r}{r!}\, \omega(f^{(r)}; D_N). \tag{5.5}$$

Remark 2. For $r \in \mathbb{N}$ the estimate (5.5) improves Corollary 4 of [3] and Corollary 3 of [2].

THEOREM 7. *Let $r \in \mathbb{N}$. Then for every function $f \in W^rB$ such that its* rth *derivative $f^{(r)}$ has concave modulus of continuity, we have*

$$|R_N^{(r)}(f)| \leqslant \frac{(D_N^{(r)})^r}{(r-1)!} \int_0^1 (1-x)^{r-1} \omega(f^{(r)}; x\, \Delta_N^{(r)})\, dx. \tag{5.6}$$

Moreover, the estimate (5.6) *cannot be improved if either r is an odd integer or r is an even integer and* (3.6) *holds.*

Proof. The estimate (5.6) follows from Theorem 2 and (5.3). Now we shall prove that this estimate is exact. From (5.2) and (4.3), we obtain

$$R_N^{(r)}(f) = \frac{1}{(r-1)!} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} (x - x_k)^r$$
$$\times \left(\int_0^1 (1-t)^{r-1} [f^{(r)}(x_k + (x - x_k)t) - f^{(r)}(x_k)]\, dt \right) dx. \tag{5.7}$$

Suppose that r is an odd integer. Let us consider again the function $\tilde{f}$ defined on $[0, 1]$ by (4.9). From (5.7) and Lemma 2, we get

$$\begin{aligned} R_N^{(r)}(\tilde{f}) &= \frac{C}{(r+1)!} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} (x-x_k)^{r+1}\,dx \\ &= \frac{C}{(r+1)!} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} |x-x_k|^{r+1}\,dx \\ &= \frac{C}{(r+1)!} \int_0^1 h(x)^{r+1}\,dx \\ &= \frac{(D_N^{(r)})^r}{(r-1)!} \int_0^1 (1-x)^{r-1} \omega(\tilde{f}^{(r)}; x\Delta_N^{(r)})\,dx. \end{aligned}$$

Hence, in this case the estimate (5.6) cannot be improved.

Now suppose that r is an even integer and (3.6) holds. Let us define the function $\bar{f}$ on $[0, 1]$ by

$$\bar{f}^{(r)}(x) = Ch(x), \tag{5.8}$$

where C is an absolute constant. From Lemma 5, it follows that

$$\begin{aligned} \omega(\bar{f}^{(r)}; \delta) &= C\delta \qquad \text{for} \quad 0 \leqslant \delta \leqslant D_N, \\ &= CD_N \qquad \text{for} \quad \delta \geqslant D_N. \end{aligned} \tag{5.9}$$

Therefore, $\bar{f} \in W^r H_1(C) \subset W^r B$ and $\omega(\delta) = \omega(\bar{f}^{(r)}; \delta)$ is a concave function. Since for every $t \in [0, 1]$, we have

$$0 \leqslant t\Delta_N^{(r)} \leqslant D_N,$$

it follows from (5.9) that

$$\omega(\bar{f}^{(r)}; t\Delta_N^{(r)}) = t\Delta_N^{(r)}.$$

Hence, from (5.7) and Lemma 2, we have

$$\begin{aligned} R_N^{(r)}(\bar{f}) &= \frac{C}{(r+1)!} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} (x-x_k)^r\,|x-x_k|\,dx \\ &= \frac{C}{(r+1)!} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} |x-x_k|^{r+1}\,dx \\ &= \frac{C}{(r+1)!} \int_0^1 h(x)^{r+1}\,dx \\ &= \frac{(D_N^{(r)})^r}{(r-1)!} \int_0^1 (1-x)^{r-1} \omega(\bar{f}^{(r)}; x\Delta_N^{(r)})\,dx. \end{aligned}$$

Consequently, in this case the estimate (5.6) cannot be improved either. Thus Theorem 7 is proved.

THEOREM 7′. *Let $r \in \mathbb{N}$ and $\omega(\delta)$ be a concave modulus of continuity. Then*

$$\sup_{f \in W^r H^\omega} |R_N^{(r)}(f)| \leqslant \frac{(D_N^{(r)})^r}{(r-1)!} \int_0^1 (1-x)^{r-1} \omega(x \Delta_N^{(r)})\, dx. \tag{5.10}$$

Besides, the inequality (5.10) *changes into equality if $\omega(\delta) = C\delta$, where C is an absolute constant.*

Proof. The estimate (5.10) follows from (5.3) and Theorem 2′. Now suppose that $\omega(\delta) = C\delta$, where C is an absolute constant. To prove the exactness of (5.10) we consider two cases. First, let r be an odd integer. Let us consider again the function $\tilde{f}$ defined on [0, 1] by (4.9). Obviously

$$\tilde{f} \in W^r H_1(C) = W^r H^\omega.$$

From (5.7) and Lemma 2, we obtain

$$\begin{aligned}
R_N^{(r)}(\tilde{f}) &= \frac{C}{(r+1)!} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} (x - x_k)^{r+1}\, dx \\
&= \frac{C}{(r+1)!} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} |x - x_k|^{r+1}\, dx \\
&= \frac{C}{(r+1)!} \int_0^1 h(x)^{r+1}\, dx \\
&= \frac{(D_N^{(r)})^r}{(r-1)!} \int_0^1 (1-x)^{r-1} \omega(x \Delta_N^{(r)})\, dx.
\end{aligned}$$

Hence, in this case the inequality (5.10) changes into equality.

Now let r be an even integer. Then let us define the function $\bar{f}$ on [0, 1] by

$$\bar{f}^{(r)}(x) = \tilde{h}(x).$$

From Lemma 4, it follows that

$$\bar{f} \in W^r H_1(C) = W^r H^\omega.$$

From (5.7) and Lemma 2, we have

$$R_N^{(r)}(\bar{f}) = \frac{C}{(r+1)!} \sum_{k=1}^{N} (x - x_k)^r \, |x - x_k| \, dx$$

$$= \frac{C}{(r+1)!} \int_0^1 h(x)^{r+1} \, dx$$

$$= \frac{(D_N^{(r)})^r}{(r-1)!} \int_0^1 (1-x)^{r-1} \omega(x \Delta_N^{(r)}) \, dx.$$

Therefore, is this case the inequality (5.10) changes into equality too. The corollary is proved.

Remark 3. Let $r \in \mathbb{N}$. From Lemma 1, it follows that

$$D_N^{(r+1)} < \Delta_N^{(r)}.$$

But it should be noted that $D_N^{(r+1)}$ cannot be placed in Theorems 7 and 7′ instead of $\Delta_N^{(r)}$.

The next theorem complements the estimate (5.4).

THEOREM 8. *For every function $f \in C$ whose modulus of continuity is a concave function, we have*

$$|R_N^{(0)}(f)| \leqslant \omega(f; D_N^{(1)}). \tag{5.11}$$

Moreover, the estimate (5.11) *cannot be improved if* (3.6) *holds.*

Proof. From (5.3) and Theorem 3, we get (5.11). Suppose that (3.6) holds. Then let us define the function $\bar{f}$ on $[0, 1]$ by

$$\bar{f}(x) = Ch(x).$$

From Lemma 5, we have

$$\begin{aligned} \omega(\bar{f}; \delta) &= C\delta \qquad \text{for} \quad 0 \leqslant \delta \leqslant D_N, \\ &= CD_N \qquad \text{for} \quad \delta \geqslant D_N. \end{aligned}$$

Therefore, $\bar{f} \in H_1(C) \subset C$ and $\omega(\delta) = \omega(\bar{f}; \delta)$ is a concave function. Since $0 \leqslant D_N^{(1)} \leqslant D_N$, it follows that

$$\omega(\bar{f}; D_N^{(1)}) = CD_N^{(1)}. \tag{5.12}$$

From (5.2), (5.12), and Lemma 2, we obtain

$$R_N^{(0)}(\bar{f}) = \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} [\bar{f}(x) - \bar{f}(x_k)]\, dx$$

$$= C \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} |x - x_k|\, dx = C \int_0^1 h(x)\, dx$$

$$= \omega(\bar{f}; D_N^{(1)}).$$

Thus Theorem 8 is proved.

From Theorems 7 and 8, we obtain the following

COROLLARY 6. *Let $r \in \mathbb{N}_0$. Then for every function $f \in W^rC$ such that its* rth *derivative $f^{(r)}$ has a concave modulus of continuity, we have*

$$|R_N^{(r)}(f)| \leqslant \frac{(D_N^{(r)})^r}{r!}\, \omega(f^{(r)}; \Delta_N^{(r)}).$$

Using Lemma 6 from Corollary 6 we get the following

COROLLARY 7. *Let $r \in \mathbb{N}_0$. Then for every function $f \in W^rC$, we have*

$$|R_N^{(r)}(f)| \leqslant 2\, \frac{(D_N^{(r)})^r}{r!}\, \omega(f^{(r)}; \Delta_N^{(r)}).$$

Remark 4. This estimate for $r = 0$ has been proved in [4].

THEOREM 8′. *We have*

$$\sup_{f \in H^\omega} |R_N^{(0)}(f)| \leqslant \omega(D_N^{(1)}). \tag{5.13}$$

Besides, the inequality (5.13) *changes into equality if $\omega(\delta) = C\delta$, where C is an absolute constant.*

Proof. The estimate (5.13) follows from (5.13) and Theorem 3′. Let $\omega(\delta) = C\delta$. Then we have from (5.2) and Lemma 2,

$$R_N^{(0)}(\tilde{h}) = \int_0^1 [\tilde{h}(x) - \tilde{h}(x_k)]\, dx$$

$$= C \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} |x - x_k|\, dx = C \int_0^1 h(x)\, dx$$

$$= \omega(\tilde{h}; D_N^{(1)}),$$

which means that (5.13) changes into equality in this case.

THEOREM 9. *Let $r \in \mathbb{N}_0$ and $0 < \alpha \leqslant 1$. Then*

$$\sup_{f \in W^r H_\alpha(C)} |R_N^{(r)}(f)| \leqslant C \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+r+1)} (D_N^{(r+\alpha)})^{r+\alpha}. \tag{5.14}$$

Moreover, the inequality (5.14) *changes into equality if either $\alpha = 1$ or r is an even integer and* (3.6) *holds.*

Proof. The estimate (5.14) follows from (5.3) and Theorem 4. If $\alpha = 1$, then the exactness of Theorem 9 can be proved as the exactness of Theorems 7′ and 8′. Now let r be an even integer and (3.6) holds. Let us define the function $\bar{f}$ on $[0, 1]$ by (4.21). Then

$$\bar{f} \in W^r H_\alpha(C).$$

From (5.7) and Lemma 2, we get for $r \in \mathbb{N}$,

$$\begin{aligned} R_N^{(r)}(\bar{f}) &= C \frac{B(\alpha+1, r)}{(r-1)!} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} (x - x_k)^r \, |x - x_k|^\alpha \, dx \\ &= C \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+r+1)} \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} |x - x_k|^{r+\alpha} \, dx \\ &= C \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+r+1)} \int_0^1 h(x)^{r+\alpha} \, dx = C \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+r+1)} (D_N^{(r+\alpha)})^{r+\alpha}. \end{aligned} \tag{5.15}$$

If $r = 0$, then from (5.2) and Lemma 2 we obtain

$$R_N^{(0)}(\bar{f}) = C \sum_{k=1}^{N} \int_{a_{k-1}}^{a_k} |x - x_k|^\alpha \, dx = C(D_N^{(\alpha)})^\alpha. \tag{5.16}$$

From (5.15) and (5.16) we conclude that in this case the inequality (5.14) changes into equality as well. Theorem 9 is proved.

Remark 5. Theorem 9 in the case $r = 0$ and $\alpha = 1$ has been proved in [14] (see p. 65, Theorem 1‴). In the case $r = 0$ and $0 < \alpha < 1$, it has been proved in [6].

THEOREM 10. *Let $r \in \mathbb{N}_0$. Then for all positive numbers p and q with $1/p + 1/q = 1$ and for every function $f \in W^r M$, we have*

$$|R_N^{(r)}(f)| \leqslant \frac{(D_N^{(pr)})^r}{r!} \tau(f^{(r)}; 2D_N)_{L_q}. \tag{5.17}$$

Proof. The estimate (5.17) follows from (5.3) and Theorem 5. Passing to the limit as $p \to \infty$ in (5.17) we obtain the following

COROLLARY 8. *Let $r \in \mathbb{N}_0$. Then for every function $f \in W^r M$, we have*

$$|R_N^{(r)}(f)| \leqslant \frac{(D_N)^r}{r!} \tau(f^{(r)}; 2D_N)_L.$$

Remark 6. This estimate for $r = 0$ has been proved in [15]. For $r \in \mathbb{N}$ it has been proved in [2, 3].

REFERENCES

1. P. D. PROINOV AND G. H. KIROV, On a quadrature formula for numerical integration in the class $C^r[0, 1]$, *C. R. Acad. Bulgare Sci.* **36** (1983), 1027–1030.
2. V. H. HRISTOV, Note on the convergence of a quadrature process, *C. R. Acad. Bulgare Sci.* **36** (1983), 1151–1154.
3. P. D. PROINOV AND G. H. KIROV, Application of uniformly distributed matrices for approximation of functions and numerical integration, *C. R. Acad. Bulgare Sci.* **37** (1984), 1625–1628.
4. G. A. TOTKOV AND M. S. BAZELKOV, On the convergence of some quadrature processes, *in* "Constructive Function Theory '84" (Proceedings, Internat. Conf., Varna, 1984), pp. 864–869, BAN, Sofia, 1984.
5. H. STEGBUCHNER, Numerische Quadratur glatter Functionen mit Gleichverteilungsmethoden, *Rend. Mat. Appl. Ser. VII* **2** (1982), 593–599.
6. P. D. PROINOV, Uniformly distributed matrices and numerical integration, *in* "Constructive Function Theory '84" (Proceedings, Internat. Conf., Varna, 1984), pp. 704–709, BAN, Sofia, 1984.
7. P. D. PROINOV, Numerical integration and approximation of differentiable functions, *C. R. Acad. Bulgare Sci.* **38** (1985), 187–190.
8. L. KUIPERS AND H. NIEDERREITER, "Uniform Distribution of Sequences," Wiley–Interscience, New York, 1974.
9. BL. SENDOV AND V. POPOV, "Averaged Moduli of Smoothness," BAN, Sofia, 1983.
10. G. PÓLYA AND G. SZEGÖ, "Aufgaben und Lehrsätze aus der Analysis," Bd. 1, Springer-Verlag, Berlin/Heidelberg, 1970.
11. N. P. KORNEICHUK, "Extremal Problems of Approximation Theory," "Nauka," Moscow, 1976.
12. P. D. PROINOV, Note on the convergence of the general quadrature process with positive weights, *in* "Constructive Function Theory '77" (Proceedings, Internat. Conf., Blagoevgrad, 1977), pp. 121–125, BAN, Sofia, 1980.
13. H. NIEDERREITER, Methods for estimating discrepancy, *in* "Applications of Number Theory to Numerical Analysis," pp. 203–236, Academic Press, New York, 1972.
14. I. M. SOBOL, "Multidimensional Quadrature Formulae and Haar's Functions," "Nauka," Moscow, 1969.
15. P. D. PROINOV, Upper bound for the error of the general quadrature process with positive weights, *C. R. Acad. Bulgare Sci.* **35** (1982), 605–608.